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"ON SOLVING BI-CRITERION MATHEMATICAL PROGRAMS"

by

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Abstract

"ON SOLVING BI-CRITERION MATHEMATICAL PROGRAMS"

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It often happens in applications of mathematical programming that there are two incommensurate objective functions to be extremized, rather than just one. One thus encounters bi-criterion programs of the form

$$\underset{x \in X}{\text{Maximize}} \quad h(f_1(x),\ f_2(x)),$$

where $h(\cdot,\cdot)$ is a (nonlinear) increasing utility indicator function defined on the possible pairs of outcomes of the concave objective functions f_1 and f_2, and x is a decision n-vector constrained to the convex set X. In this paper it is shown how such programs can be numerically solved if a parametric programming algorithm is available for the parametric sub-problem

$$\underset{x \in X}{\text{Maximize}} \quad \alpha\, f_1(x) + (1-\alpha)\, f_2(x)\ , \quad 0 \leq \alpha \leq 1.$$

Thus when the parametric sub-problem is a parametric linear program, for example, the nonlinear (and perhaps even non-concave) bi-criterion program can be solved by a modification of a stock parametric linear programming routine. An inherent advantage of the present approach is that it yields as a by-product a relevant portion of the "tradeoff curve" between f_1 and f_2. When h is quasiconcave, as is usually the case, it is shown that certain computational economies are possible. Outlines of six algorithms for solving (1) under various special assumptions are presented to illustrate the application of the theory developed herein. They are based on parametric

linear programming, Wolfe's method of parametric quadratic programming, and Geoffrion's method of parametric concave programming. Finally, two extensions are indicated: one designed to relax the convention that h must be increasing in each of its arguments rather than decreasing in one or both, and the other to permit nonlinear scale changes to be made on the f_i for convenience in solving the parametric sub-problem. These extensions greatly extend the domain of applicability and efficiency of the present approach.

ON SOLVING BI-CRITERION MATHEMATICAL PROGRAMS

In this paper we study bi-criterion programming problems of the form

$$(1) \qquad \underset{x \in X}{\text{Maximize}} \quad h(f_1(x), \quad f_2(x))$$

where f_1 and f_2 are real-valued concave (see footnote 1) criterion (payoff) functions of the n-vector x of decision variables that are constrained to lie in a convex subset X of E^n, and h is a real-valued increasing (i.e., monotone non-decreasing in each argument) ordinal utility indicator function defined on the pairs of achievable values for f_1 and f_2. We present a method for solving (1) based on any known parametric programming algorithm for the parametric sub-problem

$$(P\ \alpha) \qquad \underset{x \in X}{\text{Maximize}} \quad \alpha\, f_1(x) + (1-\alpha)\, f_2(x) \ ,$$

where the parameter α varies over the unit interval. When f_1 and f_2 are linear and X is a convex polyhedron, for example, (1) is reduced essentially to a standard parametric linear program even though h is nonlinear. Thus parametric linear programming routines can be modified to solve this important class of nonlinear (even non-concave) programs. When h is also known to be quasiconcave (i.e., there is a diminishing marginal rate of substitution between f_1 and f_2), a property shared by most utility indicator functions arising in practice, it is shown how to substantially reduce the amount of computational work necessary to solve (1). After the necessary theory is developed, it is used to construct outlines of six algorithms for solving (1) under various assumptions on f_1, f_2, and X . Two are based on parametric linear programming [5], two on Wolfe's method of parametric quadratic programming [13], and two on Geoffrion's method of parametric concave programming [6]. Finally, two extensions are presented: one designed to relax the convention that h must be increasing in each of its arguments rather than decreasing in one or both, and the other to permit nonlinear changes of scale to be made on the f_i so as to facilitate solving (P_α). These extensions enlarge the domain of successful applicability of the present approach, and lead to efficient computational algorithms for a large class of mathematical programs that can be written in the form of (1) (e.g. linear fractional programs [7,8], target assignment problems [11], and stochastic programs [10]).

Motivation

Although bi-criterion programs may be motivated in terms of consumer demand theory of classical economic analysis, the author's interest derives from a concern for an underdeveloped aspect of modern mathematical programming -- how to deal with multiple objective (criterion) functions. In practical applications of linear and nonlinear programming, alternative decisions very often have effects that cannot be naturally measured on a common scale. Consider two such effects, as represented by the criterion functions $f_1(x)$ and $f_2(x)$, and let it be desired to maximize both in the sense that, for a fixed level of one, as much as possible of the other is desired. Unless f_1 and f_2 happen to attain their maximum simultaneously on X, some compromise between the two criteria must be worked out. Five ways of proceeding are: (i) ignore one criterion and maximize the other; (ii) maximize one criterion subject to the additional constraint that the other meet or exceed some specified minimal level; (iii) maximize a weighted combination of the two criteria (without loss of generality, the result is a problem of the form (P_α) for some fixed α in the unit interval; this amounts to forcing one criterion to be measured additively in the same units as the other); (iv) determine a preference ordering over the attainable payoff set, represent it by a (usually nonlinear) utility indicator function $h(f_1,f_2)$, and solve (1); and (v) construct the admissible portion of the attainable payoff set, decide which point in it is most preferred, and determine a corresponding element of X. The attainable payoff set is the set of points $(f_1(x), f_2(x))$ in E^2 corresponding to some x in X, and the admissible portion of it (briefly, the "admissible set") consists of those points with the property that one criterion can be increased only at the expense of a decrease in the other.

Procedures (i), (ii) and (iii) are computationally the simplest, though often they lead to unrealistic reformulations of the original problem. The fourth approach can be considered a generalization of the third, and has the potential of being quite realistic. It suffers, however, from difficulties introduced by the function h. Not only must h be determined, presumably by prolonged introspection, but its composition with f_1 and f_2 can lead to a difficult maximization problem. Assuming that a computational method is available for constructing the admissible set procedure, (v) obviates the need for determining a preference ordering over the entire attainable payoff set -- for it is only necessary to determine the most preferred point of the actual admissible portion thereof arising in the context of a specific problem. An important side benefit, not automatically available under any of the other procedures, is full knowledge of the sensitivity of each criterion to the pursuit of the other. Fortunately, in a number of important cases the amount of computational work involved in computing the admissible set is not much greater than that for (iii).

We shall develop and exploit the intimate relationship between (iii), (iv), and (v) and the availability of parametric programming algorithms for (P_α) to derive a solution technique for (iv) based on computing a (usually small) part of the admissible portion of the attainable payoff set. In addition to providing a convenient computational approach, this method for solving (1) has, of course, the inherent advantage of yielding as a by-product a relevant portion of the admissible set. This suggests the following hybrid approach: make a rough determination of h (preferably having a simple analytic form), solve the resulting problem (1), and examine the by-product portion of the admissible set to see whether it provides any reason to doubt that the indicated solution is in fact the most preferred -- if so, then more of the admissible set can be computed and a revised solution chosen in the spirit of (v).

DEVELOPMENT

In addition to the assumptions stated in the first paragraph, it will be convenient to avoid questions of the attainment of suprema by assuming throughout this paper that the feasible region X is compact (closed and bounded) and non-empty as well as convex, that the f_i are continuous as well as concave[1/] on X, and that h is continuous as well as increasing on the attainable payoff set $f[X]$. We denote by f the vector-valued function (f_1, f_2), and by $f[X]$ the image in E^2 of X under f. A point $x^o \in X$ is said to be efficient if and only if there does not exist another point $x' \in X$ such that $f_i(x') \geq f_i(x^o)$, $i = 1,2$, with strict inequality holding for at least one i; in other words,

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1/ A function $f(x)$ on a convex set X is said to be concave if x^1, $x^2 \in X$, $x^1 \neq x^2$, imply $f(tx^1 + (1-t)\, x^2) \geq tf(x^1) + (1-t)\, f(x^2)$ for all $0 < t < 1$. An important property of concave functions is that a non-negative linear combination of such functions is always concave. This property is not shared by quasiconcave functions (see footnote 4).

if and only if $f(x^o)$ is in the admissible set. The set of optimal solutions of (P_α) for a fixed value of α is denoted by $X^*(\alpha)$, and any n-valued function $x^*(\alpha)$ on $[0,1]$ that satisfies $x^*(\alpha) \in X^*(\alpha)$ for each α is called an optimal solution function of (P_α).

For brevity, the basic assumptions stated above will not be explicitly mentioned in the statements of the formal lemmas and theorems below. However, the proofs will be explicit about which assumptions are actually used.

The reader will find it convenient, in following the development below, to keep Fig. 1 in mind. It represents, in payoff space, a simplified typical situation that might occur under our assumptions. The attainable payoff set is the interior of the closed curve ABCDEA, the admissible set is the heavy curve ABC, and parts of selected level curves of h have been drawn and labeled. If x^* is optimal in (1), then obviously $h(f(x^*)) = 13$ and $f(x^*)$ is point B.

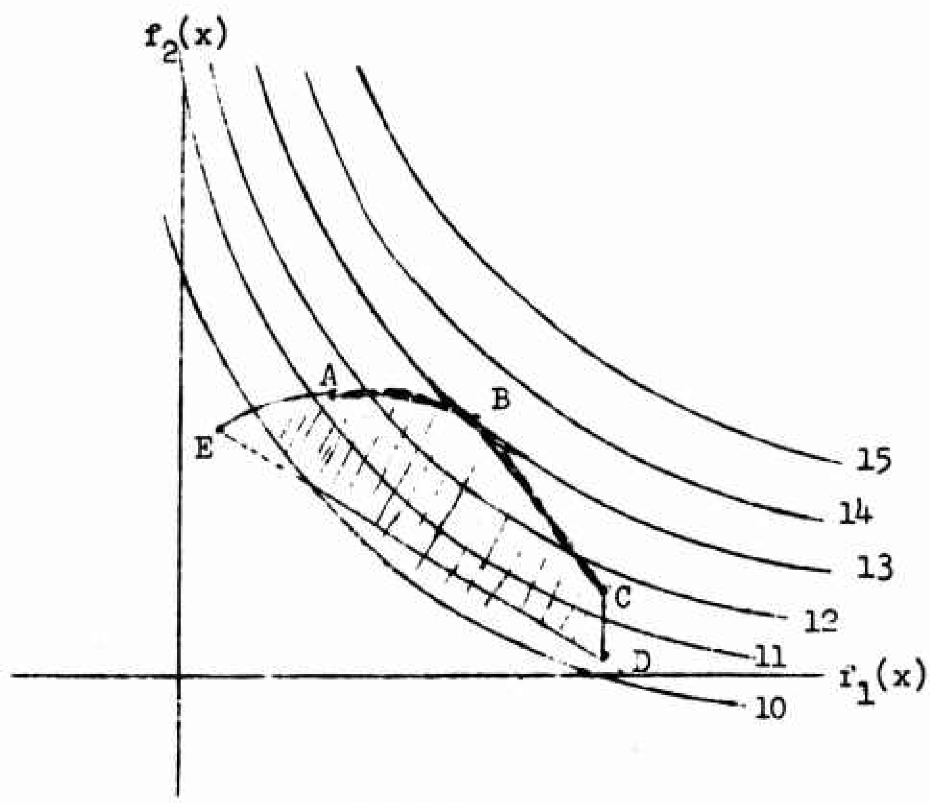

FIGURE 1

The first two lemmas provide the primary motivation for a computational approach to solving (1) in terms of the parametric program (P α). They can easily be interpreted in terms of Fig. 1.

<u>Lemma 1</u>:

At least one point at which $h(f(x))$ achieves its maximum over X is efficient.

<u>Proof</u>:

By the compactness of X and the continuity of f and h, (1) has at least one optimal solution x^o. Similarly, there exists a point x' which maximizes $f_1(x) + f_2(x)$ over X subject to the additional constraints $f_i(x) \geq f_i(x^o)$, $i = 1,2$. Now x' is easily seen to be efficient, for the contrary contradicts the choice of x'. Finally we observe that x' must also solve (1); for x' is feasible in (1), and $f_i(x') \geq f_i(x^o)$, $i = 1,2$, implies, by the fact that h is increasing, that $h(f(x')) = h(f(x^o))$.

<u>Lemma 2</u>:

If x^o is efficient, then there exists a number α^o in the unit interval such that x^o is an optimal solution of (P α^o).

<u>Proof</u>:

The proof uses the concavity of f and convexity of X in an essential way, and is an application of a basic separation property of convex sets. It can be found, for example, in Karlin [9] or Geoffrion [6].

Lemmas 1 and 2 imply

<u>Theorem 1:</u> An optimal solution of (1) is found among the optimal solutions of (P α) for some α in the unit interval. More precisely, if α^* is optimal in

(2) $$\underset{\alpha \in [0,1]}{\text{Maximize}} \quad H(\alpha) ,$$

where we define $H(\alpha)$ on the unit interval by

(3) $$H(\alpha) = \underset{x \in X^*(\alpha)}{\text{Maximum}} \quad h(f(x)),$$

then (1) is solved by any point $x \in X^*(\alpha^*)$ satisfying $h(f(x)) = H(\alpha^*)$.

That $H(\alpha)$ is well-defined follows from the non-emptiness and compactness of X and the continuity of f and h, which imply that $h(f(x))$ is continuous on the non-empty and compact set $X^*(\alpha)$. $H(\alpha)$ achieves its maximum by Lemmas 1 and 2 or by the fact that it can be shown to be an upper semi-continuous function on the compact set $[0,1]$.

The computational usefulness of (2) depends primarily on how readily $H(\alpha)$ can be computed on the unit interval. If it can be computed easily, then (2) is likely to be a quite efficient means of solving (1), for finding the maximum of $H(\alpha)$ is but a one-dimensional maximization problem. Before taking up the question of how to compute $H(\alpha)$, we point out an easy partial converse of Lemma 2 that partly justifies the assertion made earlier concerning the availability of a portion of the admissible set as a by-product of the calculations for solving (1).

Lemma 3:

Every point of $X^*(\alpha)$ is efficient when α satisfies $0 < \alpha < 1$.

Some point of $X^*(\alpha)$ is efficient when $\alpha = 0$ or 1.

Computing $H(\alpha)$

Let α be fixed in the unit interval. It might be feared that computing $H(\alpha)$ requires not only finding all optimal solutions of ($P\ \alpha$) in order to get $X^*(\alpha)$, but also solving a maximization problem of the same form as (1) itself; for it was noted that $X^*(\alpha)$ is a non-empty and compact subset of X, and from the convexity of X and the concavity of $\alpha f_1(x) + (1-\alpha) f_2(x)$ it follows that $X^*(\alpha)$ is also convex. Fortunately, however, it turns out that computing $H(\alpha)$ is not nearly so difficult as this observation would seem to indicate. The results of the following theorem show that $H(\alpha)$ can usually be computed on $[0,1]$ with little, if any, extra work beyond finding by parametric programming <u>any</u> optimal solution function $x^*(\alpha)$ of $(P\ \alpha)$ on $[0,1]$.

<u>Theorem 2:</u>

A. Let α be fixed in the unit interval. If $(P\ \alpha)$ has a <u>unique</u> optimal solution $x^*(\alpha)$, then $H(\alpha) = h(f(x^*(\alpha)))$.

B. $H(0) = h(f(x))$ for any efficient point $x \in X^*(0)$.
$H(1) = h(f(x))$ for any efficient point $x \in X^*(1)$.

C. Assume that either f_1 or f_2 (or both) is linear on a line segment in X only if it is constant on it. Then for each α satisfying $0 < \alpha < 1$, we have
$H(\alpha) = h(f(x))$ for <u>any</u> point $x \in X^*(\alpha)$.

D. Let $x^*(\alpha)$ be any optimal solution function for $(P\ \alpha)$ on $[0,1]$ that is continuous everywhere except possibly for a finite number of simple discontinuities. For each point α' of discontinuity, define $\underline{x}^*(\alpha')$ and $\bar{x}^*(\alpha')$ as the left-hand (unless $\alpha' = 0$) and right-hand (unless $\alpha' = 1$) limits of $x^*(\alpha)$ at α', respectively. Then

1. $H(\alpha) = h(f(x^*(\alpha)))$ at every point of continuity in $[0,1]$;
2. $H(0) = h(f(\bar{x}^*(0)))$ if 0 is a point of discontinuity;
3. $H(1) = h(f(\underline{x}^*(1)))$ if 1 is a point of discontinuity;

4. If α' is a point of discontinuity satisfying $0 < \alpha' < 1$, then

(3.1) $$H(\alpha') = \underset{t \in [0,1]}{\text{Maximum}} \; h(f(t \, \underline{x}^* (\alpha') + (1-t) \, \bar{x}^*(\alpha'))) \quad \text{and}$$

(3.2) $$H(\alpha') = \underset{t \in [0,1]}{\text{Maximum}} \; h(t \, f(\underline{x}^*(\alpha')) + (1-t) \, f(\bar{x}^*(\alpha'))).$$

Before undertaking to prove the results of this theorem, we remark that part A applies, for example, when $\alpha f_1(x) + (1-\alpha) f_2(x)$ is strictly concave; that part C applies, for example, if one of the f_1 is a negative semi-definite quadratic form; and that part D applies to all parametric programming algorithms known to the author in the sense that when they are applicable to $(P\ \alpha)$, they all produce an optimal solution function $x^*(\alpha)$ that is continuous everywhere on the unit interval except possibly for a finite number of simple discontinuities.[2/] Note that in part D, to compute $H(\alpha)$ for a point of discontinuity one has a choice of solving either of the two one-dimensional maximization problems (3.1) and (3.2).

The burden of the remainder of this subsection is to establish the results of Theorem 4.

Part A is trivial. Part B is nearly so, and may be seen as follows. The efficient points in $X^*(0)$ are obviously precisely those that maximize f_1 over $X^*(0)$. Since f_2 is constant over $X^*(0)$ and h is increasing, $h(f)$ is maximized over $X^*(0)$ where f_1 is maximized. Thus $H(0) = h(f(x))$ for any efficient point x in $X^*(0)$. A similar argument proves the assertion regarding $H(1)$. Toward proving parts C and D, it is convenient to write (3) in the alternate form

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2/ In fact one suspects that the exploitation of possible continuity in the optimal solution of $(P\ \alpha)$ as α varies is necessary for a successful parametric programming algorithm.

$$(4) \qquad H(\alpha) = \underset{y \in f[X^*(\alpha)]}{\text{Maximum}} \; h(y) \; ,$$

where $f[X^*(\alpha)]$ is the image of $X^*(\alpha)$ under f and y is a generic element of E^2.

Lemma 4:

For each fixed value of α satisfying $0 < \alpha < 1$, $f[X^*(\alpha)]$ is either a singleton or a compact line segment of non-zero length in E^2 with normal $(\alpha, 1-\alpha)$. In the latter case, if $f(x^1)$ and $f(x^2)$ are the end-points of the line segment, then $f(tx^1 + (1-t)x^2) = t\, f(x^1)+(1-t)\, f(x^2)$ for all t satisfying $0 \leq t \leq 1$.

Proof:

Let $0 < \alpha < 1$ be fixed. By definition, $X^*(\alpha)$ is the optimal solution set of (P_α). Hence

$$\alpha\, f_1(x) + (1-\alpha)\, f_2(x) = v(\alpha)$$

for all $x \in X^*(\alpha)$, where $v(\alpha)$ is the optimal value of (P_α). Hence $f[X^*(\alpha)]$ is a subset of the line $\{y = (y_1, y_2) \in E^2:$ $\alpha\, y_1 + (1 - \alpha)\, y_2 = v(\alpha)\}$. Suppose that $f(x^1) \neq f(x^2)$, where $x^1, x^2 \in X^*(\alpha)$. Let t be any real number in the unit interval. Then $(tx^1 + (1-t)\, x^2) \in X^*(\alpha)$ by convexity, and $f_i\, (tx^1 + (1-t)\, x^2) \geq tf_i(x^1)+ (1-t)\, f_i(x^2)$, $i = 1,2$, by concavity. Hence

$$\begin{aligned} v(\alpha) &= \alpha f_1(tx^1 + (1-t)\, x^2) + (1-\alpha)\, f_2(tx^1 + (1-t)\, x^2) \\ &\geq \alpha\, (tf_1(x^1) + (1-t)\, f_1(x^2)) \\ &\quad + (1-\alpha)\, (tf_2\, (x^1) + (1-t)\, f_2(x^2)\,) \\ &= t(\alpha\, f_1(x^1) + (1-\alpha)\, f_2(x^1)\,) \\ &\quad + (1-t)\, (\alpha f_1(x^2) + (1-\alpha)\, f_2(x^2)) \\ &= t\, v(\alpha) + (1-t)\, v(\alpha) = v(\alpha)\; , \end{aligned}$$

from which it follows (recall that $0 < \alpha < 1$) that

$$f(tx^1 + (1-t)\, x^2) = tf(x^1) + (1-t)\, f(x^2).$$

It remains only to show that $f[X^*(\alpha)]$ is compact. This follows immediately from the continuity of f and the compactness of $X^*(\alpha)$.

This lemma implies that to compute $H(\alpha)$ for fixed α satisfying $0 < \alpha < 1$ it is sufficient to know at most two points in $X^*(\alpha)$: any one point if $f[X^*(\alpha)]$ is a singleton, and any two points x^1 and x^2 which each map into a different endpoint otherwise. In the first case, from (4) we see that $H(\alpha) = h(f(x))$ for any $x \in X^*(\alpha)$; and in the second case, we have

(5) $$H(\alpha) = \underset{t \in [0,1]}{\text{Maximum}}\ h(t\, f(x^1) + (1-t)\, f(x^2))$$

or, interestingly enough, the alternative

(6) $$H(\alpha) = \underset{t \in [0,1]}{\text{Maximum}}\ h(f(tx^1 + (1-t)\, x^2)).$$

From (4), we see that the next lemma implies Th. 2C.

Lemma 5:

If either f_1 or f_2 (or both) is linear on a line segment of X only if it is constant on it, then $f[X^*(\alpha)]$ is a singleton for each fixed α satisfying $0 < \alpha < 1$.

Proof:

Let α be fixed, $0 < \alpha < 1$, and let one of the f_i, say f_1, satisfy the hypothesis. Suppose that $f[X^*(\alpha)]$ is not a singleton. Then there are points $x^1, x^2 \in X^*(\alpha)$ such that $f(x^1) \neq f(x^2)$, and by Lemma 4 $f(x)$ is linear on the line segment running from x^1 to x^2. By hypothesis, then, $f_1(x^1) = f_1(x^2)$, and so $f_2(x^1) \neq f_2(x^2)$. If $f_2(x^1) <$ (resp. $>$) $f_2(x^0)$, then x^1 (resp. x^0) is not efficient, which contradicts the known efficiency (by Lemma 3) of both x^0 and x^1.

Remark: A non-trivial example of a criterion function that is linear on a line segment of X only if it is constant on it, is a negative semi-definite quadratic form. If the other criterion function is arbitrary, say linear, then $f[X^*(\alpha)]$ must be a singleton even though $X^*(\alpha)$ need not be a singleton.

Toward proving part D of Th.2, we establish

Lemma 6:

If $x^*(\alpha)$ is any optimal solution function for $(P\alpha)$ on $[0,1]$, then $f_1(x^*(\alpha))$ (resp. $f_2(x^*(\alpha))$) is monotonically non-decreasing (resp. non-increasing) on $[0,1]$.

Proof:

Let α_1, α_2 be such that $0 \leq \alpha_1 < \alpha_2 \leq 1$. By the definitions of $x^*(\alpha_1)$ and $x^*(\alpha_2)$,

(7) $$\alpha_2 f_1(x^*(\alpha_1)) + (1-\alpha_2) f_2(x^*(\alpha_1)) \leq \alpha_2 f_1(x^*(\alpha_2)) + (1-\alpha_2) f_2(x^*(\alpha_2))$$

and

(8) $$\alpha_1 f_1(x^*(\alpha_2)) + (1-\alpha_1) f_2(x^*(\alpha_2)) \leq \alpha_1 f_1(x^*(\alpha_1)) + (1-\alpha_1) f_2(x^*(\alpha_1)).$$

Multiplying (7) by $(1-\alpha_1)$ and (8) by $(1-\alpha_2)$ and adding, after rearrangement one obtains $(\alpha_2 - \alpha_1)(f_1(x^*(\alpha_1)) - f_1(x^*(\alpha_2))) \leq 0$, from which it follows that $f_1(x^*(\alpha_1)) \leq f_1(x^*(\alpha_2))$. Similarly, by multiplying (7) by α_1 and (8) by α_2 and adding, one obtains that $f_2(x^*(\alpha_1)) \geq f_2(x^*(\alpha_2))$.

Lemma 7:

$X^*(\alpha)$ is an upper semi-continuous mapping[3/] on $[0,1]$.

3/ The definition of upper semi-continuity for set-valued functions that we use is that of Debreu [3]. As applied to $X^*(\alpha)$, upper semi-continuity at $\alpha_o \in [0,1]$ means: $<\alpha^j> \to \alpha_o$ where $\alpha^j \in [0,1]$, and $<x^*(\alpha^j)> \to x^o$, where $x^*(\alpha^j) \in X^*(\alpha^j)$, implies $x^o \in X^*(\alpha_o)$.

Proof:

Apply Th. 4 of sec. 1.8 of Debreu [3, p. 19] to (P_α) .

Lemma 8.

Let $x^*(\alpha)$ be any optimal solution function for (P_α) that is continuous on the unit interval except possibly for a finite number of simple discontinuities. At each α_0 satisfying $0 < \alpha_0 < 1$:

A. If $x^*(\alpha)$ is continuous at α_0, then $f[X^*(\alpha_0)]$ is a singleton;

B. If $x^*(\alpha)$ has a simple discontinuity at α_0, then $\underline{x}^*(\alpha_0) \equiv \lim_{\alpha \to \alpha_0 -} x^*(\alpha)$ and $\bar{x}^*(\alpha_0) \equiv \lim_{\alpha \to \alpha_0 +} x^*(\alpha)$ are both in $X^*(\alpha_0)$, and $f[X^*(\alpha_0)]$ is a compact line segment (possibly of zero length) with end points $f(\underline{x}^*(\alpha_0))$ and $f(\bar{x}^*(\alpha_0))$.

At the endpoints of the unit interval:

C. $\lim_{\alpha \to 0+} x^*(\alpha)$ is in $X^*(0)$ and is efficient;

D. $\lim_{\alpha \to 1-} x^*(\alpha)$ is in $X^*(1)$ and is efficient.

Proof:

Let $x^*(\alpha)$ be continuous at α_0 satisfying $0 < \alpha_0 < 1$. Suppose, contrary to A., that there exists $x^0 \in X^*(\alpha_0)$ such that $f(x^0) \neq f(x^*(\alpha_0))$. Since $f(x^0)$ and $f(x^*(\alpha_0))$ must both lie on a line through $f(x^0)$ with normal $(\alpha_0, 1-\alpha_0)$, either $f_1(x^0) < f_1(x^*(\alpha_0))$ or $f_1(x^*(\alpha_0)) < f_1(x^0)$.

In the first case, by the continuity of $f_1(x^*(\alpha))$ at α_0 there exists a number $\hat{\alpha}$ satisfying $\hat{\alpha} < \alpha_0$ such that $f_1(x^*(\hat{\alpha})) > f_1(x^0)$. But this contradicts the monotonicity of f_1 proved in Lemma 6. A similar contradiction can be obtained in the second case. This proves part A.

Let $x^*(\alpha)$ have a simple discontinuity at a point α_0 satisfying $0 < \alpha_0 < 1$. By Lemma 4, $f[X^*(\alpha_0)]$ is a compact line segment. Denote

$\lim_{\alpha \to \alpha_o -} x^*(\alpha)$ (resp. $\lim_{\alpha \to \alpha_o +} x^*(\alpha)$) by $\underline{x}^*(\alpha_o)$ (resp. $\bar{x}^*(\alpha_o)$). From Lemma 7, $\underline{x}^*(\alpha_o)$ and $\bar{x}^*(\alpha_o)$ are in $X^*(\alpha_o)$. It remains to show that $f(\underline{x}^*(\alpha_o))$ and $f(\bar{x}^*(\alpha_o))$ are the endpoints of $f[X^*(\alpha_o)]$. Suppose the contrary. Then there exists $x^o \in X^*(\alpha_o)$ such that $f_1(x^o) < f_1(\underline{x}^*(\alpha_o))$ and $f_2(x^o) > f_2(\underline{x}^*(\alpha_o))$, or $f_1(\bar{x}^*(\alpha_o)) < f_1(x^o)$ and $f_2(\bar{x}^*(\alpha_o)) > f_1(x^o)$. We shall consider the first case and construct the contradiction that there exists a value of α such that

$$\alpha\, f_1(x^o) + (1-\alpha)\, f_2(x^o) > \alpha\, f_1(x^*(\alpha)) + (1-\alpha)\, f_2(x^*(\alpha)).$$

A similar construction leads to a contradiction for the second case.

For all $\alpha \in (o,1)$, we have $\alpha(f_1(x^o) - f_1(x^*(\alpha)))$

$$+ (1-\alpha)(f_2(x^o)-f_2(x^*(\alpha))) = (\alpha - \alpha_o + \alpha_o)\,(f_1(x^o) - f_1(x^*(\alpha)))$$

$$+ (1-\alpha + \alpha_o - \alpha_o)\,(f_2(x^o)-f_2(x^*(\alpha))) = (\alpha - \alpha_o)\,[f_1(x^o) - f_1(x^*(\alpha))$$

$$+ f_2(x^*(\alpha)) - f_2(x^o)] + \{\alpha_o(f_1(x^o) - f_1(x^*(\alpha))) + (1-\alpha_o)(f_2(x^o) - f_2(x^*(\alpha)))\}$$

$$\geq (\alpha-\alpha_o)\,[f_1(x^o) - f_1(x^*(\alpha)) + f_2(x^*(\alpha)) - f_2(x^o)\,],$$

where the last inequality follows from the fact that the quantity in curly brackets is non-negative (recall that x^o solves $(P\alpha_o)$). By the left continuity of $f(x^*(\alpha))$ at α_o and the fact that $(f_1(x^o) - f_1(\underline{x}^*(\alpha_o)))$ and $(f_2(\underline{x}^*(\alpha_o)) - f_2(x^o))$ are both negative, the desired inequality is established for all α less than but sufficiently near α_o. This completes the proof of part B.

Finally we prove part C. A similar argument proves part D. By Lemma 7, $\bar{x}^*(0) \equiv \lim_{\alpha \to 0+} x^*(\alpha)$ is in $X^*(0)$. Suppose that $\bar{x}^*(0)$ is not efficient. Then there exists a point $x^o \in X$ such that

$f_1(x^o) > f_1(\bar{x}^*(0))$ and $f_2(x^o) = f_2(\bar{x}^*(0))$ (since $\bar{x}^*(0)$ solves (P_o), $f_2(x^o) > f_2(\bar{x}^*(0))$ is impossible). Thus $x^o \in X^*(0)$. Let $\hat{\alpha} > 0$ be such that $f_1(x^o) > f_1(x^*(\hat{\alpha}))$. This contradicts the monotonicity of f_1 established in Lemma 6.

Th. 2D is established by parts A and B of Lemma 8 in conjunction with Lemma 4, and parts C and D of Lemma 8 in conjunction with Th. 2B.

The case in which h is quasiconcave

In this sub-section we introduce an additional hypothesis on the utility indicator function h which permits attention to be restricted to a (hopefully small) subinterval of [0,1] when (2) is being executed. We assume now that h is quasiconcave[4/] on the convex hull[5/] F of the admissible payoff set. Quasiconcavity is a weaker property than concavity, and is almost universally assumed as a property of utility indicator functions in consumer demand theory of traditional economic analysis. For further discussion of quasiconcavity, see Arrow and Enthoven [1].

An immediate consequence of this additional hypothesis, in the presence of our previous assumptions, is that $h(f(x))$ is now quasiconcave on X (see e.g, Berge [2, p. 207]). Although (1) now becomes susceptible to various direct (non-parametric) approaches to quasiconcave programming, the approach represented by Th. 1 can be very efficient when an efficient parametric programming algorithm is available for $(P\alpha)$ -- especially in view of Theorem 3 below.

4/ $h(y)$ is quasiconcave on the convex set F if and only if $\{y \in F: h(y) \geq k\}$ is a convex set for all real k. An equivalent definition is that $h(t\, y^1 + (1-t)\, y^2) \geq \text{Min}\, \{h(y^1), h(y^2)\}$ for all y^1, y^2 in F and $0 < t < 1$. Simple examples of quasiconcave increasing h are: $\text{Min}\{y_1, y_2\}$; $y_1 \cdot y_2$ for $y_1, y_2 \geq 0$; and $y_1^{\lambda_1} \cdot y_2^{\lambda_2}$ for $\lambda_1, \lambda_2 \geq 0$ and $y_1, y_2 > 0$.

5/ The convex hull of a subset of Euclidean space is the smallest convex set containing that set.

Lemma 9:

Let x^i solve $(P\alpha^i)$, $i = 0,1,2$, where $0 \leq \alpha^1 < \alpha^0 < \alpha^2 \leq 1$ Then there exists a number t, $0 \leq t \leq 1$, such that

$$f_i(x^0) \geq t\, f_i(x^1) + (1-t)\, f_i(x^2), \quad i = 1,2.$$

Proof:

Denote $f(x^i)$ by f^i, $i = 0,1,2$. We assume that f^0 does not coincide with either f^1 or f^2, for otherwise the conclusion of the lemma would be trivially true. Suppose that the conclusion is false. Then there does not exist a number $t \geq 0$ that satisfies the following system of inequalities

(9.1) $\quad t(f_1^1 - f_1^2) \leq (f_1^0 - f_1^2)$

(9.2) $\quad t(f_2^1 - f_2^2) \leq (f_2^0 - f_2^2)$

(9.3) $\quad t \leq 1.$

By a standard theorem on non-negative solutions to linear inequalities [4, p. 47], there exist non-negative real numbers s_1, s_2, and s_3 such that

(10) $\quad (f_1^1 - f_1^2)\, s_1 + (f_2^1 - f_2^2)\, s_2 + s_3 \geq 0$

and

(11) $\quad (f_1^0 - f_1^2)\, s_1 + (f_2^0 - f_2^2)\, s_2 + s_3 < 0.$

Multiplying (11) by -1 and adding the result to (10), one obtains

(12) $\quad (f_1^1 - f_1^0)\, s_1 + (f_2^1 - f_2^0)\, s_2 > 0.$

Using the fact that $s_3 \geq 0$, from (11) one obtains

(13) $\quad (f_1^2 - f_1^0)\, s_1 + (f_2^2 - f_2^0)\, s_2 > 0.$

Now s_1 and s_2 cannot both vanish. Dividing (12) and (13) by $(s_1 + s_2)$, recalling that $s_1, s_2 \geq 0$, and defining ξ as $s_1/(s_1 + s_2)$, one obtains

(14) $\quad (f_1^1 - f_1^0)\xi + (f_2^1 - f_2^0)\,(1 - \xi) > 0,$

(15) $\quad (f_1^2 - f_1^0)\xi + (f_2^2 - f_2^0)\,(1 - \xi) > 0,$

and $0 \leq \xi \leq 1$.

Define $v_j(\alpha) \equiv \alpha f_1^j + (1 - \alpha) f_2^j$, $j = 0,1,2$.

By the definitions of x^i, $= 0,1,2$, $v_j(\alpha^j) \geq v_k(\alpha^j)$ for $j = 0,1,2$ and $k \neq j$. Thus

(16) $v_1(\alpha^1) - v_o(\alpha^1) \geq 0$

(17) $v_1(\alpha^o) - v_o(\alpha^o) \leq 0$

(18) $v_2(\alpha^2) - v_o(\alpha^2) \geq 0$

(19) $v_2(\alpha^o) - v_o(\alpha^o) \leq 0$.

Now (14) and (15) may be written as

(20) $v_1(\xi) - v_o(\xi) > 0$.

(21) $v_2(\xi) - v_o(\xi) > 0$.

By the linearity of $v_1(\alpha) - v_o(\alpha)$ in α, (16), (17), and (20) imply that $\xi < \alpha^o$ (recall that $\alpha^1 < \alpha^2$). Similarly, (18), (19), and (21) imply that $\xi > \alpha^o$. This contradiction implies that the conclusion of the lemma must be true.

<u>Theorem 3</u>:

Assume that h is quasiconcave on F. If $x^*(\alpha)$ is <u>any</u> optimal solution function of $(P\alpha)$ on $[0,1]$, then $h(f(x^*(\alpha)))$ is quasiconcave on $[0,1]$.

<u>Proof:</u>

Let $0 \leq \alpha^1 < \alpha^o < \alpha^2 \leq 1$, and let $x^i \in X^*(\alpha^i)$, $i = 0,1,2$. By Lemma 9, there exists a number t, $0 \leq t \leq 1$, such that

$f_i(x^o) \geqq t\, f_i(x^1) + (1-t)\, f_i(x^2)$, $i = 1,2$.

Thus $h(f(x^o)) \geq h\,(t\, f(x^1) + (1-t)\, f(x^2))$

$\geq \text{Min}\, \{h\,(f(x^1)), h(f(x^2))\}$, where the first inequality holds because h is increasing and the second because it is quasiconcave. This shows that $h(f(x^*(\alpha)))$ is quasiconcave on $[0,1]$.

Theorem 3 often makes possible a considerable simplification in the maximization of $H(\alpha)$ on the unit interval, by allowing part of the interval to be ignored by virtue of the easy

Corollary 3.1:

If $x^1 \in X^* (\alpha^1)$ and $x^2 \in X^*(\alpha^2)$, where $\alpha^1 < \alpha^2$, then $h(f(x^1)) - h(f(x^2)) < 0$ (resp. > 0) implies that $H(\alpha)$ cannot achieve its maximum at $\alpha < \alpha^1$ (resp. $\alpha > \alpha^2$).

EXEMPLARY ALGORITHMS

In this section we apply the results of the last (principally Part D of Th. 2 and Cor. 3.1) to show how known parametric programming algorithms can be used to solve (1) in the manner suggested by Th. 1. For illustrative purposes we choose parametric linear programming [5], Wolfe's method of parametric quadratic programming [13], and Geoffrion's method of parametric concave programming [6]. The six algorithms presented below are given in outline form, with no attempt made to give details of the most efficient organization of the computations.

Parametric Linear Programming

In this subsection we assume that f_1 and f_2 are linear and that X is determined by linear inequality constraints, so that parametric linear programming can be used to produce an optimal solution function $x^*(\alpha)$ for $(P\alpha)$ on $[0,1]$. It is well known that $x^*(\alpha)$ will be piecewise constant, and that without loss of generality it can be assumed to be of the form

$$x^*(\alpha) = x^i \quad \text{for} \quad \alpha^i \leq \alpha < \alpha^{i+1}, \; i = 0,\ldots, N,$$

where $0 < \alpha^1 < \ldots < \alpha^N < 1$ (N finite and possibly 0) are the points of discontinuity and we have put $\alpha^0 = 0$ and $\alpha^{N+1} = 1$. Also, $x^*(1) = x^N$. Thus by Th. 2D we have $H(\alpha) = h(f(x^i))$ for $\alpha^i < \alpha < \alpha^{i+1}$, $i = 0,\ldots,N$, $H(0) = h(f(x^0))$, and $H(1) = h(f(x^N))$. If $N = 0$, then obviously x^0 is optimal in (1). If $N \geq 1$, then we have $\underline{x}(\alpha^i) = x^{i-1}$ and $\bar{x}^*(\alpha^i) = x^i$ for $i = 1,\ldots, N$; consequently, (3.1) and (3.2) become

$$(22) \qquad H(\alpha^i) = \underset{t\in[0,1]}{\text{Maximum}} \; h(f(t\, x^{i-1} + (1-t)\, x^i))$$

$$(23) \qquad H(\alpha^i) = \underset{t\in[0,1]}{\text{Maximum}} \; h(t\, f(x^{i-1}) + (1-t)\, f(x^i))$$

for $i = 1,\ldots, N$. Since $x^*(\alpha)$ is piecewise constant we see that when $N \geq 1$, $H(\alpha)$ achieves its maximum at a point of discontinuity α^{i*}; therefore the point $t^*\, x^{i*-1} + (1-t^*)\, x^{i*}$ is optimal in (1), where t^* satisfies $H(\alpha^{i*}) = h(f(t^*\, x^{i*-1} + (1-t^*)\, x^{i*}))$ or, alternatively, $H(\alpha^{i*}) = h(t^*\, f(x^{i*-1}) + (1-t^*)\, f(x^{i*}))$ (cf. (5) and (6)). We thus obtain the following algorithm.

<u>Algorithm 1</u>

Step 1. Solve $(P\alpha)$ by parametric linear programming to obtain α^i and x^i, $i = 0,\ldots,N$, computing the quantities $H(\alpha^i)$, $i = 1,\ldots, N$ by (22) or (23) as the calculations progress.

If $N = 0$, stop: x^0 is optimal in (1). If $N \geq 1$, then go to step 2.

Step 2. Let $H(\alpha^{i*})$ be the largest of the quantities computed at step 1. Then $t^* x^{i*-1} + (1-t^*) x^{i*}$ is optimal in (1), where t^* is defined as in the text so as to achieve $H(\alpha^{i*})$. Stop.

If h is quasiconcave, then due to the consequent quasiconcavity of $H(\alpha)$ it is rarely necessary to solve $(P\alpha)$ on the entire unit interval, or to compute all of the $H(\alpha^i)$. In Algorithm 2, which exploits the quasiconcavity of h, it is assumed for simplicity of exposition that the parameter α increases, starting from the value 0. A similar algorithm can easily be constructed to cover the more general case in which α has an arbitrary starting value and can decrease as well as increase (the closer the starting value is to the one that maximizes $H(\alpha)$, the less work is required to solve (1) by this approach). This same remark applies to Algorithms 4 and 6.

Algorithm 2

Step 1. Solve (P_0) to obtain x^0. Put $I = 0$ and $\underline{I} = 1$.

Step 2. Solve $(P\alpha)$ by parametric linear programming as α increases above α^I until either $\alpha = 1$ or α^{I+1} is encountered. In the first case, go to step 4; in the second, determine x^{I+1} and go to step 3.

Step 3. Compare $h(f(x^I))$ with $h(f(x^{I+1}))$:

a. If $h(f(x^I)) < h(f(x^{I+1}))$, increase I by 1, put $\underline{I} = I$, and return to step 2.

b. If $h(f(x^I)) = h(f(x^{I+1}))$, increase I by 1 and return to step 2;

c. If $h(f(x^I)) > h(f(x^{I+1}))$, increase I by 1 and go to step 4.

Step 4. If $I = 0$, stop; x^0 is optimal in (1). If $I \geq 1$, then compute by (22) or (23) and find the largest of the quantities $H(\alpha^i)$ $(\underline{I} \leq i \leq I)$. If the maximum is achieved for $H(\alpha^{i}*)$, then $t^* x^{i_* -1} + (1-t^*) x^{i}*$ is optimal in (1), where t^* is defined as above so as to achieve $H(\alpha^{i}*)$. Stop.

Remark:

In both of these algorithms, a one-dimensional maximization problem ((22) or (23)) must be solved each time an $H(\alpha^i)$ is required. Frequently these one-dimensional problems are trivial; but even when they are not, various methods are available [12]. When h is quasiconcave, Fibonnacci search is particularly attractive.

Parametric Quadratic Programming

In this subsection we assume that $f_1(x)$ is linear, that $f_2(x)$ is a negative semi-definite quadratic form, and that X is determined by linear inequality constraints. Then $(P\alpha)$ can be solved on $[0,1]$ by Wolfe's method of parametric quadratic programming (his so-called "long form") [13], among others, for an optimal solution function $x^*(\alpha)$ that is continuous on $[0,1]$. By Th. 2.D.1, $H(\alpha) = h(f(x^*(\alpha)))$ on $[0,1]$, and therefore the point x^* in the image of $[0,1]$ under $x^*(\alpha)$ which maximizes $h(f(x))$ is also optimal in (1). Now from Wolfe's results it follows easily that this image set is of the form $\bigcup_{i=0}^{N} \overline{x^i,x^{i+1}}$, where $\overline{x^i,x^{i+1}}$ is a line segment in E^n with endpoints x^i and x^{i+1} and N is a finite positive integer. The points $x^i (i = 0,1,\ldots,N + 1)$ are determined serially, in order of increasing superscript, from the modified Simplex procedure employed by Wolfe 6/ ($x^i \equiv x^*(\alpha^i)$ for certain α^i satisfying $o = \alpha^o < \alpha^1 < \ldots < \alpha^N < \alpha^{N+1} = 1$); a termination signal accompanies the determination of x^{N+1}. Putting these observations together, we obtain

Algorithm 3

Step 1. Solve $(P\alpha)$ on $[0,1]$ by Wolfe's method to obtain x^i, $i = 0,1,\ldots,N + 1$, computing the quantities

$$\eta^i \equiv \underset{0\leq\lambda\leq 1}{\text{Maximum}} \; h(f(\lambda\, x^{i-1} + (1-\lambda)\, x^i)),\; i = 1,\ldots,N+1,$$

as the calculations proceed.

Step 2. If η^{i*} is the largest of the η^i (ties are immaterial) then $\lambda^*\; x^{i*-1} + (1-\lambda^*)\; x^{i*}$ is optimal in (1), where λ^* satisfies $\eta^{i*} = h(f(\lambda^*\; x^{i*-1} + (1-\lambda^*)\; x^{i*}))$.

If h is quasiconcave, then so is $h(f(x^*(\alpha)))$, and an improved version of Algorithm 3 can be constructed that bears much the same relation to it as Algorithm 2 does to Algorithm 1:

Algorithm 4

Step 1. Solve (Po) and obtain x^o by Wolfe's method. Put $\underline{I} = 1$ and $I = 0$.

6/ Actually, Wolfe's algorithm is addressed to a reparameterized version of $(P\alpha)$ that uses $\lambda/\lambda+1$ on $[0,\infty]$ in place of α on $[0,1]$. But this causes no essential difficulty.

Step 2. If $I = N + 1$, go to step 4; otherwise, determine x^{I+1} by Wolfe's method and go to step 3.

Step 3. Compare $h(f(x^I))$ with $h(f(x^{I+1}))$:

a. If $h(f(x^I)) < h(f(x^{I+1}))$, increase I by 1, put $\underline{I} = I$, and return to step 2;

b. If $h(f(x^I)) = h(f(x^{I+1}))$, increase I by 1 and return to step 2;

c. If $h(f(x^I)) > h(f(x^{I+1}))$, increase I by 1 and go to step 4.

Step 4. Compute the quantities $\eta^i \equiv \underset{0 \le \lambda \le 1}{\text{Maximum}} \; h(f(\lambda x^{i-1} + (1-\lambda)\, x^i))$, $\underline{I} \le i \le I$. If η^{i*} is the largest (ties are immaterial), then $\lambda^* x^{i_*-1} + (1-\lambda)\, x^{i_*}$ is optimal in (1), where λ^* satisfies $\eta^{i*} = h\,(f(\lambda^* x^{i_*-1} + (1-\lambda)\, x^{i_*}))$.

The remark following Algorithm 2 is appropriate here also with regard to computing the η^i, especially when h is quasiconcave ... for then $h(f(\lambda\, x^{i-1} + (1-\lambda)\, x^i))$ is quasiconcave in λ on $[0,1]$.

Parametric Concave Programming

When X is determined by concave inequality ($\geq$) constraints and certain additional hypotheses are satisfied, Geoffrion's method [6] can be used to solve $(P\alpha)$ on $[0,1]$. The $x^*(\alpha)$ so produced is continuous, and so by Th. 2.D.1. it follows that $H(\alpha) = h(f(x^*(\alpha)))$ on $[0,1]$.

Algorithm 5

Solve $(P\alpha)$ on $[0,1]$ by Geoffrion's method to obtain $x^*(\alpha)$, all the while evaluating $h(f(x^*(\alpha)))$. Determine the value α^* at which $h(f(x^*(\alpha)))$ achieves its maximum on $[0,1]$. Then $x^*(\alpha^*)$ is optimal in (1).

If h is quasiconcave, then as before $(P\alpha)$ does not ordinarily have to be solved on the entire unit interval:

Algorithm 6

Step 1. Solve (P_0) by Geoffrion's or some other method to obtain $x^*(0)$.

Step 2. Determine $x^*(\alpha)$ as α increases above 0 by Geoffrion's method, all the while evaluating $h(f(x^*(\alpha)))$, until a value α^* is encountered above which $h(f(x^*(\alpha)))$ begins to decrease. Then $x^*(\alpha^*)$ is optimal in (1).

Algorithms 5 and 6 are not limited to Geoffrion's method, of course, but apply equally well to any parametric concave programming algorithm that produces a continuous optimal solution $x^*(\alpha)$ to $(P\alpha)$ on $[0,1]$.

EXTENSIONS

In mathematical programming with one objective function, the convention is usually made to discuss only maximization problems or only minimization problems, for the results for one class of problems are directly applicable to the other when appropriate sign changes are made. The same situation prevails here. We have chosen to discuss the case where both f_i are being maximized, but our results are applicable to the other cases (min f_1, min f_2; min f_1, max f_2; max f_1, min f_2) with appropriate sign changes. If h is increasing in f_1 but decreasing in f_2, for example, define f_2' as $-f_2$ and $h'(f_1,f_2')$ as $h(f_1,-(-f_2))$. Then h' is increasing in both f_1 and f_2', and if f_2' is concave (this is true if and only if f_2 is a convex function) then our results apply if h' is used in place of h and f_2' is used in place of f_2. As an example of the application of this idea, consider the "linear fractional" programming problem.7/

$$\underset{x \geqq 0}{\text{Maximize}} \quad (cx + \gamma)/(dx + \delta) \text{ subject to } Ax \leqq b, \text{ where } c \text{ and } d$$

are n-vectors, b is an m-vector, A is an m x n matrix, and γ and δ are scalars. We assume for simplicity that $f_1(x) \equiv cx + \gamma$ and $f_2(x) \equiv dx + \delta$ are strictly positive for all feasible x. Defining $h \equiv f_1(x)/f_2(x)$, we observe that h is increasing in $f_1(x)$ but decreasing in $f_2(x)$. Thus we consider h' and f_2', defined as above, in place of h and f_2. Since it is easily seen that h' is quasiconcave and that f_1 and $f_2' \equiv (-f_2)$ are linear, Algorithm 2 applies, thereby providing a procedure for solving the linear fractional program by means of parametric

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7/ The linear fractional programming problem is due to Isbell and Marlow [7, p. 82]. Several methods for solving such programs are available, most of them based on linear programming techniques. For a brief guide to the literature, see Joksch [8, p. 197].

linear programming.[8/] Note that (22) and (23), which are used by Algorithm 2, are particularly simple in this case. If $dx + \delta$ is replaced by a positive semi-definite quadratic form, then by similar reasoning we see that Algorithm 4 provides a method for "linear/quadratic fractional" programming.

A trick sometimes useful in ordinary mathematical programming is to perform a nonlinear change of scale on the criterion function in order to make it concave (assuming that it is to be maximized) or of a simpler functional form, so that an available algorithm can be applied. For example, to minimize $\Pi_{i=1}^{n} q_i^{x_i}$ over $x \geqq 0$ satisfying $Ax \leqq b$, where $q_i > 0$ ($i = 1,\ldots,n$), it is more convenient to minimize $\ln(\Pi_{i=1}^{n} q_i^{x_i}) = \Sigma_{i=1}^{n} x_i(\ln q_i)$ instead because then linear programming can be used. In the remainder of this section we shall show that this idea can be used to greatly extend the power of the present method of bi-criterion programming.

Define the scale-modified parametric sub-problem

$$(\tilde{P}\alpha) \qquad \underset{x \in X}{\text{Maximize}} \ \alpha\, g_1(f_1(x)) + (1-\alpha) g_2(f_2(x)),$$

where the g_i are henceforth assumed to be strictly increasing, differentiable functions defined at least on $f_i[X]$. We shall also assume that $g_i(f_i(x))$ is concave (a sufficient but not necessary condition for which is that g_i be concave) and for simplicity that the f_i are differentiable. We denote the optimal solution set of $(\tilde{P}\,\alpha)$ by $\tilde{X}(\alpha)$.

In the sequel h is not assumed to be quasiconcave except in Theorem 3A. We shall obtain counterparts of Lemmas 2 and 3 and Theorems 1, 2, and 3.

<u>Lemma 2A:</u>

If x^o is efficient, then there exists a number α^o in the unit interval such that x^o is an optimal solution of $(\tilde{P}\,\alpha_o)$.

8/ The assumption that $(f_1(x), f_2(x))$ is in the interior of the positive orthant for all feasible x can be relaxed, at the expense of slightly modifying Algorithm 2, to the minimal requirement that $dx + \delta \neq 0$ for all feasible x.

Proof:

Since x^o is efficient, by Lemma 2 there exists a number v in the unit interval such that x^o is an optimal solution of (Pv). Because of the concavity of $vf_1(x) + (1-v)\, f_2(x)$,
we therefore have

(24) $\nabla_x (v\, f_1(x) + (1-v)\, f_2(x))\,(x-x^o) \leq 0, \quad \forall\, x \in X,$

where ∇_x is the gradient operator. Put

$$\alpha_o = \frac{(v/g_1'(f_1(x^o)))}{(v/g_1'(f_1(x^o))) + ((1-v)/g_2'(f_2(x^o)))},$$

where $g_i'(f_i(x^o))$ is the first derivative of g_i evaluated at $f_i(x^o)$. Since g_i is strictly increasing, we have $g_i'(f_i(x^o)) > 0$, $i = 1,2$. Hence α_o is in the unit interval. To show that x^o is optimal in $(\tilde{P}\,\alpha_o)$, because of the concavity of $\alpha_o g_1(f_1(x)) + (1-\alpha_o)\, g_2(f_2(x))$ on X it is equivalent to show

(25) $\nabla_x\, (\alpha_o g_1(f_1(x^o)) + (1-\alpha_o)\, g_2(f_2(x^o)))\,(x-x^o) \leq 0, \; \forall\, x \in X.$

By the definition of α_o, the gradient vector in (25) is

$$(1/\beta)\ \{(v/g_1'(f_1(x^o)))\, g_1'(f_1(x^o))\, \nabla_x f_1(x^o) +$$

$$((1-v)/\, g_2'(f_2(x^o)))\, g_2'(f_2(x^o))\, \nabla_x f_2(x^o)\},$$

where $\beta = (v/g_1'(f_1(x^o))) + ((1-v)/g_2'(f_2(x^o)))$.

Upon cancelling and observing that $\beta > 0$, we see that (24) follows from (25).

Lemmas 1 and 2A imply

Theorem 1A:

An optimal solution of (1) is found among the optimal solutions of $(\tilde{P}\,\alpha)$ for some α in the unit interval. More precisely, if α^* is optimal in

$$\text{(2A)} \qquad \underset{\alpha \in [0,1]}{\text{Maximize}} \quad \tilde{H}(\alpha) \ ,$$

where we define

$$\text{(3A)} \qquad \tilde{H}(\alpha) \equiv \underset{x \in \tilde{X}(\alpha)}{\text{Maximum}} \quad h(f(x)),$$

then (1) is solved by any point $x \in \tilde{X}(\alpha^*)$ satisfying $h(f(x)) = \tilde{H}(\alpha^*)$.

We also have

<u>Lemma 3A</u>:

<u>Every</u> point of $\tilde{X}(\alpha)$ is efficient when α satisfies $0 < \alpha < 1$. <u>Some</u> point of $\tilde{X}(\alpha)$ is efficient when $\alpha = 0$ or 1.

To know how to compute $\tilde{H}(\alpha)$ economically, we require the following version of Th. 2.

<u>Theorem 2A</u>:

A. Let α be fixed in $[0,1]$. If $(\tilde{P}\ \alpha)$ has a <u>unique</u> optimal solution $\tilde{x}(\alpha)$, then $\tilde{H}(\alpha) = h(f(\tilde{x}(\alpha)))$.

B. $\tilde{H}(0) = h(f(x))$ for <u>any</u> <u>efficient</u> point $x \in \tilde{X}(0)$.
$\tilde{H}(1) = h(f(x))$ for <u>any</u> <u>efficient</u> point $x \in \tilde{X}(1)$.

C. Assume that $g_1(f_1(x))$ or $g_2(f_2(x))$ or both is linear on a line segment of X only if it is constant on it. Then for each α satisfying $0 < \alpha < 1$, we have $\tilde{H}(\alpha) \equiv h(f(x))$ for <u>any</u> point $x \in \tilde{X}(\alpha)$.

D. Let $\tilde{x}(\alpha)$ be any optimal solution function for $(\tilde{P}\ \alpha)$ on $[0,1]$ that is continuous everywhere except possibly for a finite number number of simple discontinuities. For each point α' of discontinuity, define $\underset{=}{\tilde{x}}(\alpha')$ and $\bar{\tilde{x}}(\alpha)$ as the left-hand (unless $\alpha' = 0$) and right-hand (unless $\alpha' = 1$) limits of $\tilde{x}(\alpha)$ at α', respectively. Then

1. $\tilde{H}(\alpha) = h(f(\tilde{x}(\alpha)))$ at every point of continuity in $[0,1]$.
2. $\tilde{H}(0) = h(f(\bar{\tilde{x}}(0)))$ if 0 is a point of discontinuity.
3. $\tilde{H}(1) = h(f(\underset{=}{\tilde{x}}(0)))$ if 1 is a point of discontinuity.
4. If α' is a point of discontinuity satisfying $0 < \alpha' < 1$, then

$$(3.1A)\quad \tilde{H}(\alpha') = \underset{t \in [0,1]}{\text{Maximum}}\; h(f(t\,\underset{=}{\tilde{x}}(\alpha') + (1-t)\,\bar{\tilde{x}}(\alpha'))) \quad \text{and}$$

$$(3.2A)\quad \tilde{H}(\alpha') = \underset{t \in [0,1]}{\text{Maximum}}\; h(g^{-1}(tg(f(\underset{=}{\tilde{x}}(\alpha'))) + (1-t)\,g(f(\bar{\tilde{x}}(\alpha'))))).$$

Proof:

The proof of this theorem will not be given in detail here, inasmuch as it follows closely that of Th. 2. The key observation is that $(\tilde{P}\,\alpha)$ has all of the properties that $(P\,\alpha)$ does, if we view $g(f)$ in $(\tilde{P}\,\alpha)$ as taking the place of f in $(P\,\alpha)$. Thus Lemmas 4 through 8 hold with regard to $(\tilde{P}\,\alpha)$ if their statements are modified by replacing everywhere *- superscripts by tildes and f by $g(f)$. To relate the results regarding $(\tilde{P}\,\alpha)$ to $h(f(\tilde{x}(\alpha)))$, it is necessary to observe that $h(f(x)) \equiv h(g^{-1}(g(f(x))))$, where $g^{-1} = (g_1^{-1}, g_2^{-1})$. The (single-valued) inverse functions g_i^{-1} exist because the g_i are strictly increasing.

Theorem 3A:

Assume that h is quasiconcave on F. If $\tilde{x}(\alpha)$ is any optimal solution function of $(\tilde{P}\,\alpha)$ on $[0,1]$, then $h(f(\tilde{x}(\alpha)))$ is quasiconcave on $[0,1]$.

Proof:

Let α^1, α^0, and α^2 satisfy $0 \leq \alpha^1 < \alpha^0 < \alpha^2 \leq 1$, and let x^i be in $\tilde{X}(\alpha^i)$, $i = 0,1,2$. If $f(x^0) = f(x^1)$ or $f(x^0) = f(x^2)$, then obviously $h(f(x^o)) \geq \text{Min}\ \{h(f(x^1)),\ h(f(x^2))\}$. We shall show that this conclusion holds when $f(x^o) \neq f(x^1)$ and $f(x^o) \neq f(x^2)$, thereby showing that $h(f(\tilde{x}(\alpha)))$ is quasiconcave on $[0,1]$.

If $0 < \alpha^1 < \alpha^o < 1$, then x^1 and x^o are efficient and by Lemma 2 there exist λ^1, λ^o in the unit interval such that $x^i \in X^*(\lambda^i)$, $i = 0,1$. It follows from Lemma 6A, which implies that $g(f(\tilde{x}(\alpha)))$ is monotone in α, the fact that $f(x^o) \neq f(x^1)$, the strictly increasing nature of g, and $\alpha^1 < \alpha^o$, that $\lambda^1 < \lambda^o$. If $\alpha^1 = 0$, then clearly $x^1 \in X^*(0)$. Thus for $0 \leq \alpha^1 < \alpha^o < 1$ we have the existence of λ^1 and λ^2 satisfying $0 \leq \lambda^1 < \lambda^o \leq 1$ such that $x^i \in X^*(\lambda^i)$, $i = 0,1$. By similar arguments we obtain that for $0 < \alpha^o < \alpha^2 \leq 1$ there exists λ^2 satisfying $\lambda^o < \lambda^2 \leq 1$ such that $x^2 \in X^*(\lambda^2)$. Applying Lemma 9, we find that there exists a number t, $0 \leq t \leq 1$, such that $f_i(x^o) \geqq t f_i(x^1) + (1-t)\, f_i(x^2)$, $i = 1,2$. Hence

$$\begin{aligned} h(f(x^o)) &\geq h(t\, f(x^1) + (1-t)\, f(x^2)) \\ &\geq \text{Min}\ \{h\, (f(x^1)),\ h(f(x^2))\}, \end{aligned}$$

where the first inequality holds because h is increasing and the second because h is quasiconcave. The proof is complete.

Theorems 1A, 2A, and 3A can be used in the same manner as were Theorems 1, 2, and 3 to construct algorithms for solving (1) via $(\tilde{P}\ \alpha)$. The freedom to perform nonlinear scale changes on the f_i can be used to extend the applicability of the known parametric programming algorithms.

As an example, consider the problem

(26) $$\underset{x \geqq 0}{\text{Minimize}} \quad v_1 \, \Pi_{j=1}^{n} \, q_{1j}^{x_j} + v_2 \, \Pi_{j=1}^{n} \, q_{2j}^{x_j}$$

$$\text{subject to } Ax \leqq b \ ,$$

where the q_{ij} and v's are strictly positive. Problems of this sort arise in redundancy allocation and target - assignment contexts. While (26) can be approximated as a linearly separable convex program by an appropriate change of variables [11, p. 350], we shall indicate how it can be solved via parametric linear programming. By making the obvious identifications

$$H = f_1 + f_2$$

$$f_i(x) = v_i \, \Pi_{j=1}^{n} \, q_{ij}^{x_j} \ , \quad i = 1,2$$

$$X = \{x \geqq 0 \, : \, Ax \leqq b\} \ ,$$

we see that (26) is a problem of the form (1). The obvious scale change to apply to the f_i is the logarithmic one: $g_i(y) \equiv - \ln(-y)$, $i = 1,2$. Theorems 1A, 2A, and 3A then apply, and Algorithm 2 can be used to solve (1) by parametric linear programming applied to the <u>linear</u> sub-problem

(27) $$\underset{x \geqq 0}{\text{Maximize}} \quad \alpha \, \Sigma_{j=1}^{n} \, x_j \, (-\ln q_{1j}) + (1-\alpha) \sum_{j=1}^{n} x_j (-\ln q_{2j})$$

$$\text{subject to } Ax \leqq b.$$

The stochastic programming model of Kataoka [10] can be solved by a scale-change that leads to the applicability of Algorithm 4.

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13 ABSTRACT

In this paper we study bi-criterion programming problems of the form

$$\text{(1)} \qquad \underset{x \in X}{\text{Maximize}} \quad h(f_1(x), \; f_2(x)),$$

where f_1 and f_2 are concave criterion functions of the n-vector x of decision variables that are constrained to lie in a non-empty convex subset X of E^n, and h is an increasing ordinal utility indicator function defined on the pairs of achievable values for f_1 and f_2. We present solution techniques based on known parametric programming algorithms. When f_1 and f_2 are linear and X is a convex polyhedron, for example, (1) is reduced essentially to a standard parametric linear program even though h is nonlinear. Outlines of six algorithms under various special assumptions are presented to illustrate the application of the theory developed herein. Some extensions are presented that extend the domain of applicability and efficiency of the present approach.

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